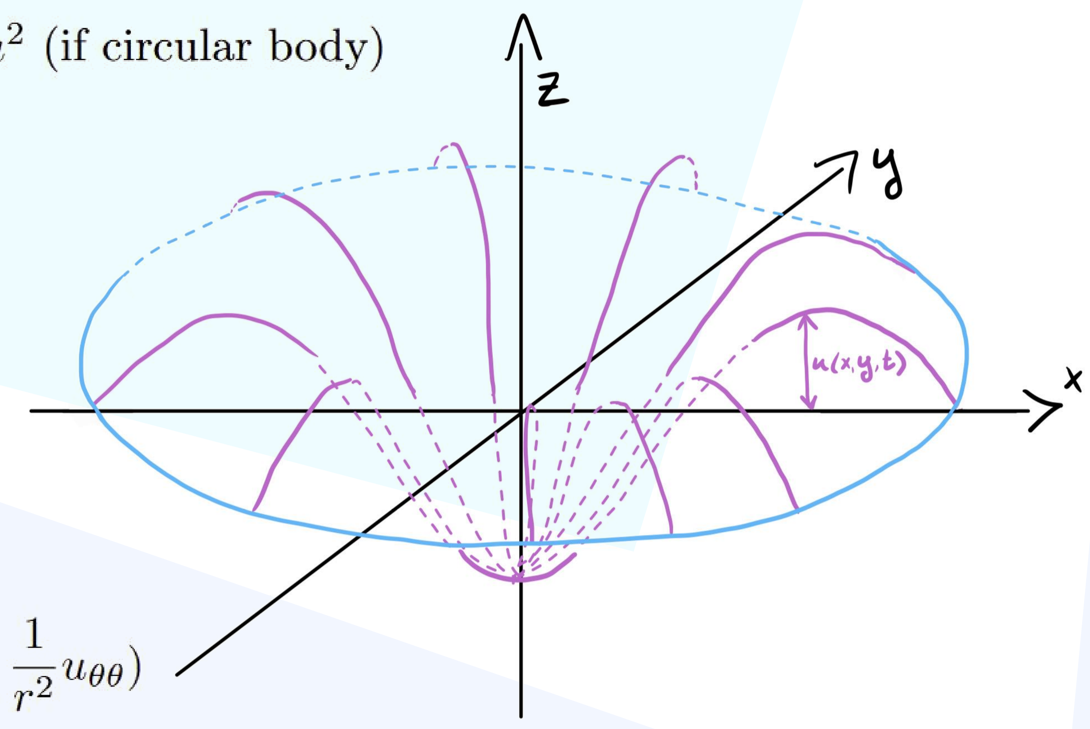
Vibrations of a drumhead

Ismael Parada
ismaelp@kth.se
Elias Lundell
ellundel@kth.se
logflames.github.io

1 Introduction

We study the vibrations of a drumhead, modelled as a circular membrane attached around a fixed border, we also later explore numerical methods for non-circular membranes. The extension u follows the wave equation and we get the following system:

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}) \text{ on } x^2 + y^2 \le a^2 \text{ (if circular body)} \\ u = 0 \text{ on border} \\ u(x, y, 0) = u_0(x, y) \end{cases}$$



2 Solution

2.1 Derivation of solution

We convert it to polar coordinates,

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$$

and separate the variables,

$$u(r, \theta, t) = T(t)R(r)\Theta(\theta) \implies$$

$$\frac{T''}{c^2T} = \frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta}.$$

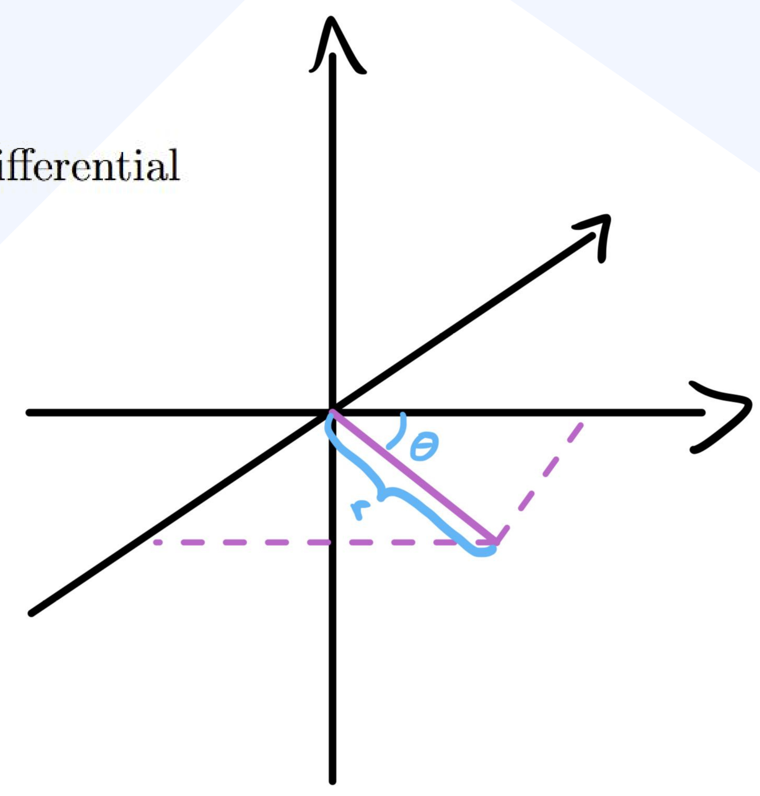
Seeing that $\frac{T''}{c^2T}$ and $\frac{\Theta''}{r^2\Theta}$ must be independent of r, θ and t we conclude they are constant. By setting them to $-\lambda$ and $-\gamma$ and get the following system,

$$\begin{cases} T'' + c^2\lambda T = 0 \\ \Theta'' + \gamma\Theta = 0 \\ R'' + \frac{1}{r}R' + (\lambda - \frac{\gamma}{r^2}R) = 0. \end{cases}.$$

The first two equations are second order homogeneous differential equations which are easily solvable.

$$T(t) = A\sin(c\lambda t) + B\cos(c\lambda t)$$
$$\Theta(\theta) = C\sin(\sqrt{\gamma}\theta) + D\cos(\sqrt{\gamma}\theta)$$

From $\Theta(\theta) = \Theta(\theta + 2\pi)$ we get that $\gamma = n^2$.



2.2 Bessel's equation

The third equation is a bit more difficult. But by scaling it by $\sqrt{\lambda}$ and introducing $\rho = \sqrt{\lambda}r$ we get *Bessel's equation* of base n.

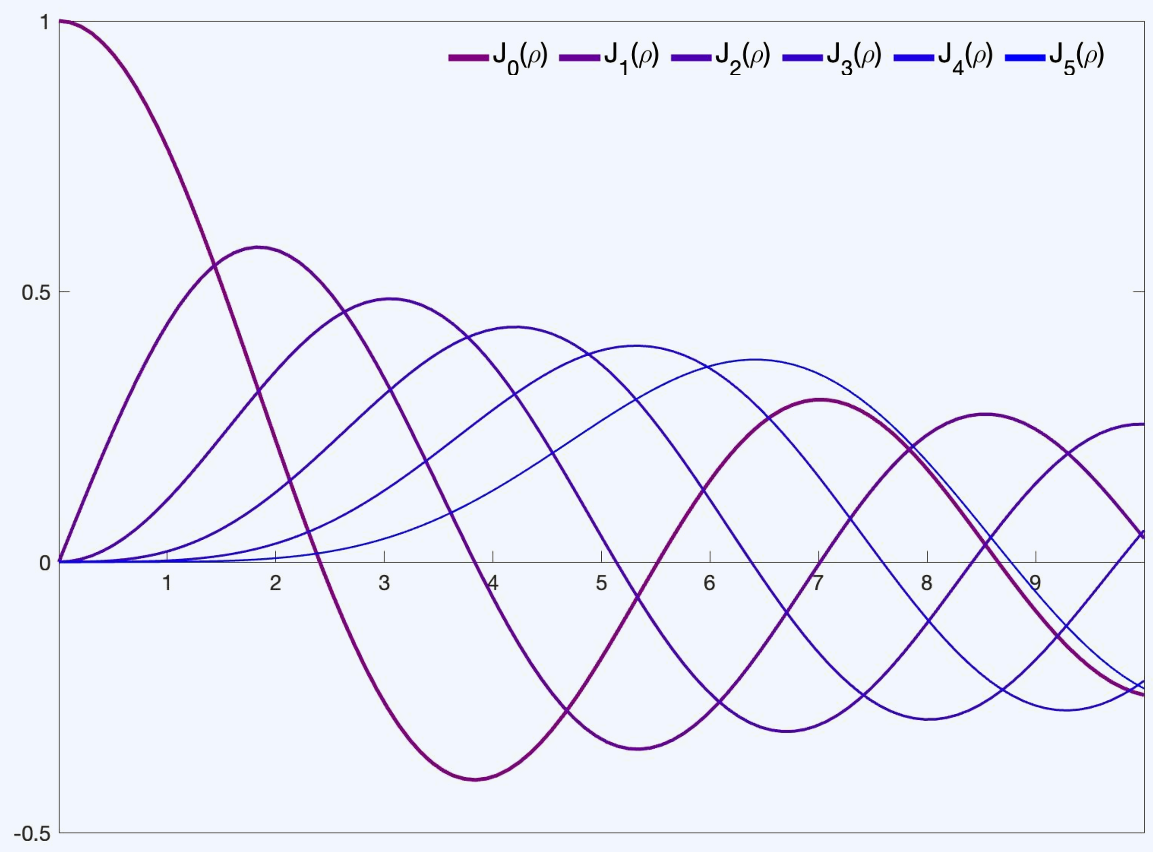
$$R_r = R_\rho\frac{d\rho}{dr} = \sqrt{\lambda}R_\rho, \quad R_{rr} = \lambda R_{\rho\rho} \implies$$

$$R_{\rho\rho} + \frac{1}{\rho}R_\rho + (1 - \frac{n^2}{\rho^2})R = 0.$$

Bessel's equation cannot be solved directly, but can be solved as an infinite series.

$$J_n(\rho) = \sum_{j=0}^{\infty}(-1)^j\frac{\frac{1}{2}\rho^{n+2j}}{j!\,(n+j)!}.$$

Plotted for $n = 0, \ldots, 5$ we have:



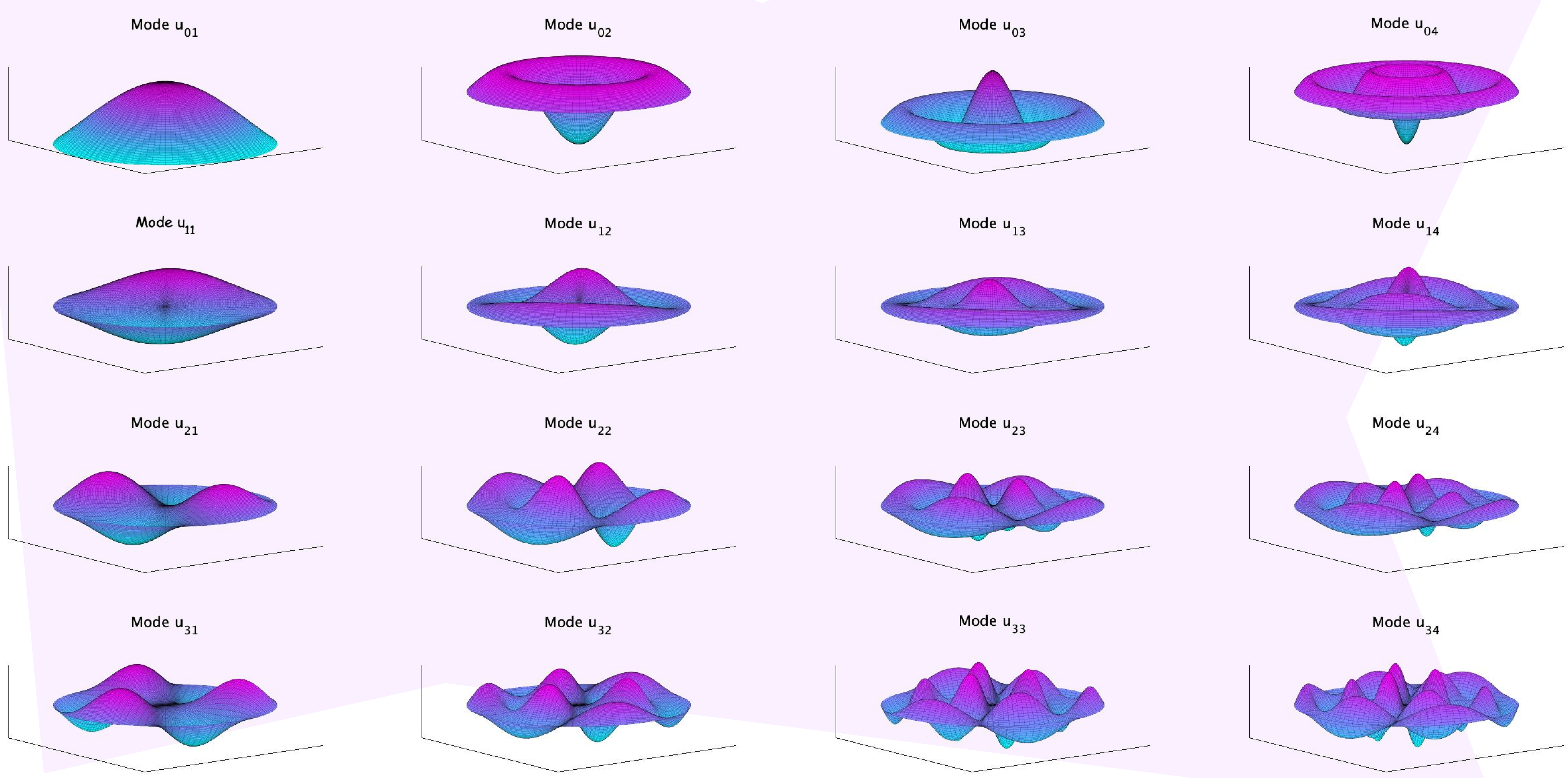
2.3 Solution

Put together, we get the full general solution

$$u(r,\theta,t) = \sum_{m=1}^{\infty} J_0(\sqrt{\lambda_{0m}}r)(A_{0m}\cos(\sqrt{\lambda_{0m}}ct) + C_{0m}\sin(\sqrt{\lambda_{0m}}ct)) + \sum_{m,n=1}^{\infty} J_n(\sqrt{\lambda_{nm}}r)\Big((A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta))\cos(\sqrt{\lambda_{nm}}ct) + (C_{nm}\cos(n\theta) + D_{nm}\sin(n\theta))\sin(\sqrt{\lambda_{nm}}ct)\Big)$$

3 Modes

We study the modes of the drum, which all vibrations can be broken down into in our infinite series. These are derived from the roots of Bessel's equation. Let λ_{nm} be the mth root of J_n, $\gamma_n = n^2$. We let $a = 1$, $c = 100$ and $A = B = C = D = 1$ and get the following plot.



4 Non-circular drums

So far, we have only explored circular drums. Trying to solve these equations for non-circular drums is very difficult, and was for a long time unsolved. We wanted to explore the behaviour of non-cirular drums and choose to do so using PDEtool in Matlab. The results can be seen in the following QR-codes.

Circular Drum

Star-shaped Drum

[1] Strauss W.A. "Partial Differential Equations An Introduction", 2nd Edition
[2] Wikipedia. "Vibrations of a circular membrane", available at: https://en.wikipedia.org/wiki/Vibrations_of_a_circular_membrane